STRONGLY COUPLED QUANTUM DISCRETE LIOUVILLE THEORY. I: ALGEBRAIC APPROACH AND DUALITY

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ABSTRACT. The quantum discrete Liouville model in the strongly coupled regime, $1 < c < 25$, is formulated as a well defined quantum mechanical problem with unitary evolution operator. The theory is self-dual: there are two exponential fields related by Hermitean conjugation, satisfying two discrete quantum Liouville equations, and living in mutually commuting subalgebras of the quantum algebra of observables.

INTRODUCTION

The Liouville equation[33]

$$(1) \qquad \phi_{tt} - \phi_{xx} - 4e^{-2\phi} = 0$$

has plenty of important applications in Mathematics and Physics. In particular, it describes the surfaces of constant negative curvature and plays the indispensable role in uniformization theory of Riemannian surfaces [35] (for some recent approach see [40, 41, 42]).

In modern physics the Liouville equation defines one parameter family of models in conformal field theory (CFT), which usually is identified with 2-dimensional gravity [30]. It plays even more important role in Polyakov's theory of noncritical Bosonic string in dimensions $d < 26$ [36]. For these reasons the Liouville model, especially in its quantum version, attracted wide attention during the last 25 years [10, 11, 25, 26].

The parameter c, labelling the quantum Liouville theory as a CFT model, is the central charge in a representation of the Virasoro algebra. The quasiclassical (or weak coupling) region, corresponding to large positive c, is well understood. The domain $c < 1$, containing the minimal models of CFT for some preferred discrete values of c, is also well described [9, 22, 18]. It is the region $1 < c < 25$, corresponding to strong coupling, about which there exists very limited knowledge until now.

In this paper we begin to describe one more method to treat the Liouville model, which is applicable for studying the strong coupling region. The method is based on the apparatus of quantum integrable models (see e. g. [14] for the recent survey). Some parts of this machinery were already used in CFT and, in particular, for the Liouville model. However, we feel that we have added several new things to this development.

First, we use the lattice regularization for the model, which exactly retains the integrability. Here we follow the previous papers [16, 21]. Second, we show that it is indispensable (especially in the strong coupling region)

to use simultaneously two mutually dual discrete models, corresponding to two exponents of the coupling constant $\tau^{\pm 1}$, symmetrically entering the expression for the central charge

$$c = 1 + 6(\tau + \tau^{-1} + 2).$$

Positive τ correspond to $c > 25$ or weak couplings, while strong couplings lead to complex τ on unit circle,

$$\tau = e^{i\theta}, \quad \tau^{-1} = e^{-i\theta} = \bar{\tau},$$

and only unification of the models for τ and $1/\tau$ can restore unitarity. Similar considerations on constructing "modular doubles" were used earlier in simpler examples of the Weil–Heisenberg algebra [13] and quantum group [15]. This paper, technically being more involved, ideologically belongs to the same line of thought. We must stress, that one can see the elements of the dualization idea in papers [1, 24, 10, 43] devoted to the Liouville model itself, and in [5, 6, 7, 9] within the framework of conformal field theory.

Our main result consists in constructing the unitary evolution operator for the chiral shift serving the both dual models.

In the first part of the paper we remind various facts about the Liouville equation to be used in what follows. Many of these facts are known in one or another form; however we present them in the form most suitable for our goal. Then follows description of the discretized Liouville equation and its formal quantization in purely algebraic manner. The appropriate involution together with proper quantization are introduced in section 4. The self-dual structure is indispensable for that.

1. Recollecting the facts

In this section we remind in appropriate form some facts about the Liouville model and the technique of its discretization.

1.1. Liouville formula and Möbius-invariance.
Liouville formula makes solutions, all of them in fact, from pairs of arbitrary functions of one variable "moving" in light-cone directions

$$(2) \qquad -e^{-2\phi(x,t)} = \frac{\alpha'(x-t)\beta'(x+t)}{(\alpha(x-t) - \beta(x+t))^2}.$$

The right hand side of eqn (2) is invariant under simultaneous point-wise Möbius transformations

$$\alpha \mapsto \alpha \triangleleft M = \frac{m_{21} + \alpha m_{11}}{m_{22} + \alpha m_{12}} \qquad \beta \mapsto \beta \triangleleft M$$

of the "chiral halves" α and β. Accordingly, if those are "Möbius-periodic"

$$\alpha(x + 2\pi) = \alpha(x) \triangleleft T \qquad \beta(x + 2\pi) = \beta(x) \triangleleft T,$$

with the same monodromy matrix T in both chiralities of course, the solution comes out periodic in the spatial direction

$$\phi(x + 2\pi, t) = \phi(x, t).$$

From now on, this will be the only boundary condition in use.

Because of this hidden Möbius-symmetry, the chiral halves cannot be uniquely restored from a given solution, but their Schwarz derivatives can, for they are Möbius-invariant as well. In particular, that of α (times minus one half)

$$u = -\tfrac{1}{2}\left(\tfrac{\alpha'''}{\alpha'} - \tfrac{3}{2}(\tfrac{\alpha''}{\alpha'})^2\right)$$

doubles as the (chiral component of the) stress-energy tensor

(3) $$u(x - t) = \tfrac{1}{4}(\phi_x - \phi_t)^2 + \tfrac{1}{2}(\phi_{xx} - \phi_{tx}) + e^{-2\phi}.$$

We will now pay less attention the parallel, or rather perpendicular, β-chirality.

1.2. Magri bracket.

The canonical Poisson bracket

$$\{\varpi(x), \varphi(y)\} = \delta(x - y)$$

on the laboratory Cauchy data

$$\varphi = \phi|_{t=0} \qquad\qquad \varpi = \phi_t|_{t=0},$$

together with canonical total momentum and energy

$$P = \int \varpi \varphi' dx \qquad\qquad H = \tfrac{1}{2}\int (\varpi^2 + \varphi'^2 + 4e^{-2\varphi}) dx$$

gives the equations of motion in the Hamiltonian formalism:

$$\varphi' = \{P, \varphi\} \qquad\qquad \dot{\varphi} = \varpi = \{H, \varphi\}$$
$$\varpi' = \{P, \varpi\} \qquad\qquad \dot{\varpi} = \varphi'' + 4e^{-2\varphi} = \{H, \varpi\}.$$

These equations lead to a single free motion equation

(4) $$\dot{u} = -u' = \{L_0, u\},$$

with L_0 denoting the (slightly shifted) zero'th Fourier coefficient

$$L_0 = \int (u + \tfrac{1}{4})\, dx = \tfrac{1}{2}(H - P + \pi).$$

The bracket involved is the same canonical bracket of course, but now we are prompted to express it, using formula (3) at $t = 0$, in terms of either u itself or its Fourier coefficients $L_a = \int (u + \tfrac{1}{4})e^{-iax}dx$. The Magri bracket [34]

(5) $$\{u(x), u(y)\} = (u(x) + u(y))\, \delta'(x - y) - \tfrac{1}{2}\delta'''(x - y)$$

and yet more celebrated (Poisson bracket realization of) the Virasoro algebra

$$-i\{L_a, L_b\} = (a - b)L_{a+b} + \pi(a^3 - a)\delta_{a,-b}$$

emerge. They will attract our attention in the next couple of (sub)sections where we shall try to treat the Magri bracket and the "chiral Hamiltonian" L_0 as a stand-alone dynamical system, not for the first time of course. We shall review some earlier attempts to get it properly discretized and

quantized. It will give an idea how far we can go on bare common sense and formal algebra, without actually touching the Liouville equation.

1.3. **Volterra model.** The first such attempt [16] produced a viable lattice counterpart of the Magri bracket

$$
(5') \qquad \begin{aligned} \{u_m, u_n\} = \tfrac{1}{4}\, u_m u_n\big((4 - u_m - u_n)(\delta_{m+1,n} - \delta_{m-1,n}) \\ - u_{\frac{1}{2}(m+n)}(\delta_{m+2,n} - \delta_{m-2,n}), \end{aligned}
$$

with $u_n \sim 1 - \Delta^2 u(n\Delta)$. In the traditional difference-differential scheme of things it has to do with Volterra's venerable preys and predators

$$
(4') \qquad \dot{u}_n = u_n(u_{n-1} - u_{n+1}) = \left\{ \sum \log u_m, u_n \right\},
$$

otherwise known as the lattice KdV equation. It should be noted however that the latter only emerges in a rather tricky continuous limit, while the simplest limit leads to the free equation (4). Which is a little unfortunate, for however essential a part has the Volterra system been playing in the soliton theory, there must be something wrong about a nonlinear equation emulating a linear one. Apparently, discrete space and continuous time do not get along, still, the KdV connection may teach us a thing or two.

1.4. **Bi-hamiltonity.** Historically, the Magri bracket brought about the notion of "raising and lowering" which soon became paramount for the whole Hamiltonian theory of soliton equation. In our case, one lifts L_0 to

$$
L_0^{\uparrow} = \int \tfrac{1}{4}\, u^2 dx
$$

to turn the free motion (4) into the KdV equation

$$
\dot{u} = \tfrac{1}{2} u''' - 3uu' = \{L_0^{\uparrow}, u\},
$$

then drops the Magri bracket to that of Zakharov-Faddeev's

$$
\{u(x), u(y)\}_{\downarrow} = 2\delta'(x - y)
$$

to restore the free motion in a different guise

$$
\dot{u} = -u' = \{L_0^{\uparrow}, u\}_{\downarrow}.
$$

Although not quite relevant to the Liouville equation, this makeshift may be useful anyway, for the corresponding lower guise of the Volterra system

$$
\dot{u}_n = u_n(u_{n-1} - u_{n+1}) = \left\{ \sum u_m, u_n \right\}_{\downarrow}
$$

features a more quantization-friendly bracket

$$
\{u_m, u_n\}_{\downarrow} = u_m u_n(\delta_{m+1,n} - \delta_{m-1,n}).
$$

We happen to already know [19] how to turn this one into a noncommutative algebra and to arrange there a "free motion in discrete time". So, in the rest of this section we shall be looking at that motion, then we shall try to guess how it could be lifted back to the Magri–Virasoro case.

1.5. **Quantization.** Going quantum, bracket $_\downarrow$ ought to become Weyl-style exchange relations

$$u_m u_n = q^{2(\delta_{m+1,n} - \delta_{m-1,n})} u_n u_m \ .$$

We do not want to discuss before time the nature of quantization constant q or the exact degree of formality implied in what follows, but we want to be very clear about boundary conditions: the u's are strictly periodic and so is the Kronecker symbol of course

$$u_{n+N} = u_n \qquad\qquad \delta_{m,n} = \begin{cases} 1 & \text{if } m \equiv n \pmod{N} \\ 0 & \text{otherwise,} \end{cases}$$

on top of that, for the reason to surface shortly, we want period N to be even and an additional condition

(6) $$u_1 u_3 \dots u_{N-1} = u_2 u_4 \dots u_N$$

to be met, it makes sense because the elements in both sides are central. With all this in place, the following relation

$$\hat{u}_n = u_{n-1} = Q^{-1} u_n Q,$$

with

$$Q = b_1 b_2 \dots b_{N-1} \qquad\qquad b_n = \sum_{a=-\infty}^{\infty} q^{a^2} u_n^a \ ,$$

proves to hold for all $n \in \mathbb{Z}$. So, with automorphism $\hat{\ } : u_n \mapsto u_{n-1}$ interpreted as a jump in time, the above relation replaces the (lower) Volterra system with a free and fully discrete Heisenberg quantum motion, employing the element Q as an "evolution operator".

1.6. **Proof.** We just honestly start from $n = 1$:

$$Q^{-1} u_1 Q = Q^{-1} u_1 b_1 b_2 b_3 \dots b_{N-1} = Q^{-1} b_1 \sum_{a=-\infty}^{\infty} q^{a^2+2a} u_2^a u_1 b_3 \dots b_{N-1}$$

$$= Q^{-1} b_1 b_2 q^{-1} u_2^{-1} u_1 b_3 \dots b_{N-1} = Q^{-1} b_1 b_2 b_3 u_2^{-1} u_1 u_3 b_4 \dots b_{N-1} = \cdots$$

— every step adds another factor, in the end parity of N and that center-reducing condition (6) make all the difference —

$$\cdots = Q^{-1} Q (u_2 u_4 \dots u_{N-2})^{-1} u_1 u_3 \dots u_{N-1} = u_N.$$

We may seem to have done only one very particular case, but, luckily, the rest is little more than tautology. It immediately follows that $Q^{-1} b_1 Q = b_N$, which can be written as

$$Q = b_1^{-1} Q b_N = b_2 b_3 \dots b_N,$$

which in turn just means that $Q = \hat{Q}$ and therefore anything good for $n = 1$ at once becomes just as good for all n. That is it.

In the process, a seeming contradiction between the "open-ended" appearance of the evolution operator and the cyclic nature of its action has resolved itself. Since

$$Q = b_1 b_2 \dots b_{N-1} = \hat{Q} = b_2 b_3 \dots b_N = \hat{\hat{Q}} = \cdots = b_N b_{N+1} \dots b_{N-2} \ ,$$

whichever of these forms one chooses to express Q in, it always remains an ordered product but each time starts from another point, in this sense it does not depend on the starting point, or rather does not have one. The next subsection offers an explanation of this miracle.

1.7. **It has to do with braids.** Let us compile a list of what we know about the b's. They are periodic, $b_{n+N} = b_n$, they are this "local"

$$b_m b_n = b_n b_m \qquad \text{if } |m-n| \not\equiv 1 \pmod{N},$$

and they satisfy "global" relations which closed the previous subsection

$$b_1 b_2 \ldots b_{N-1} = b_2 b_3 \ldots b_N = \cdots = b_N b_{N+1} \ldots b_{N-2} .$$

Let us now consider these as defining relations and identify the emerging group. It is B_N, the group of braids of N strings in 3 dimensions, alternatively defined by an exhaustive list of (Artin's) relations

$$b_n b_{n-1} b_n = b_{n-1} b_n b_{n-1} \qquad\qquad b_n b_m = b_m b_n \text{ if } |n-m| \neq 1$$

imposed on $N-1$ generators $b_1, b_2, \ldots, b_{N-1}$. We leave it as an (instructive) exercise to check that our list is equivalent to Artin's, provided the leftmost of our "global" relations is read as

$$b_N = (b_2 \ldots b_{N-1})^{-1} b_1 b_2 \ldots b_{N-1}$$

and understood as the definition of b_N. Just in case, let us warn against mistaking b_N for the N'th generator of B_{N+1}, ours is just an element of B_N explicitly defined above. Those familiar with the braid group must have already recognized in it the first string crossing the very last one behind all the others, let us picture it for $N = 4$:

$$b_1 = \text{} \qquad b_2 = \text{} \qquad b_3 = \text{} \qquad b_4 = \text{} .$$

A picture of the evolution operator

$$Q = b_1 b_2 b_3 = b_2 b_3 b_4 = b_3 b_4 b_1 = b_4 b_1 b_2 = \text{} ,$$

now explains better than words how it manages to be ordered and cyclic at the same time.

1.8. **Lattice Virasoro algebra.** Now we look for a generalization of this construction by natural interpretation of the "raising". So, we want the lattice Virasoro algebra actually to be a group. If that is to be, the "global" relations ought to remain the same as in the braid group

$$d_1 d_2 \ldots d_{N-1} = d_2 d_3 \ldots d_N = \cdots = d_N d_{N+1} \ldots d_{2N-2} ,$$

or equivalently

$$(4'') \qquad \hat{d}_n = d_{n-1} = Q^{-1} d_n Q \qquad\qquad Q = d_1 d_2 \ldots d_{N-1} ,$$

but this time each generator should interfere not only with the nearest neighbours but also with the second-nearest ones

$$d_m d_n = d_n d_m \qquad \text{if } |m-n| \not\equiv 1 \text{ and } 2 \pmod{N},$$

just like it was in bracket (5$'$) of course. This by itself implies

$$d_n d_{n-2} d_{n-1} d_n d_{n+1} = d_{n-2} d_{n-1} d_n d_{n+1} d_{n-1}$$

instead of Artin's relations, but we feel that these are not tight enough, so we voluntarily split each of them in two

$$d_n d_{n-2} d_{n-1} d_n d_{n+1} = d_{n-2} d_n d_{n-1} d_{n+1} = d_{n-2} d_{n-1} d_n d_{n+1} d_{n-1},$$
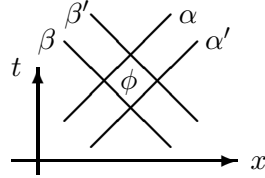
to end up with weird relations from [47]

(5$''$)
$$d_{n+1} d_{n-1} d_n d_{n+1} = d_{n-1} d_{n+1} d_n$$
$$d_n d_{n-1} d_{n+1} = d_{n-1} d_n d_{n+1} d_{n-1}.$$

This coup de force will find its justification in the treatment of the Liouville model below.

2. Difference-difference Liouville equation

2.1. **Liouville formula.** For aesthetical reason alone, its difference approximation can only be

$$-\Delta^2 e^{-2\phi} \sim \frac{(\alpha' - \alpha)(\beta' - \beta)}{(\alpha' - \beta')(\alpha - \beta)},$$

with primes now denoting finite shifts of arguments by Δ. Appearance aside, we have already mentioned that the Liouville formula manifests invariance of ϕ under simultaneous point-wise Möbius transform of chiral halves α and β. The so called cross-ratio employed in our difference scheme has just the same property.

To make of this a lattice formula we draw a square j, k lattice and put

$$\chi_{jk} = -\frac{\left(\alpha_{\frac{1}{2}(j-k+1)} - \alpha_{\frac{1}{2}(j-k-1)}\right)\left(\beta_{\frac{1}{2}(j+k+1)} - \beta_{\frac{1}{2}(j+k-1)}\right)}{\left(\alpha_{\frac{1}{2}(j-k+1)} - \beta_{\frac{1}{2}(j+k+1)}\right)\left(\alpha_{\frac{1}{2}(j-k-1)} - \beta_{\frac{1}{2}(j+k-1)}\right)}$$

in the vertices with $j + k \equiv 1 \pmod 2$, like this:
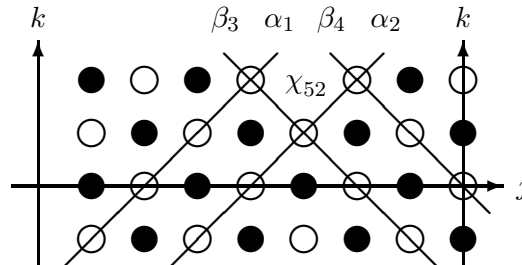
So, the χ's occupy sites marked by bullets, the empty circlets will this time remain empty. The second k-axis at $j = 8$ reminds that the lattice covers a cylinder rather than a plane, $\chi_{j+2N,k} = \chi_{jk}$, with N not necessarily equal four of course.

2.2. **Discrete Liouville equation.** It not only ideologically justifies the chosen discretization but also helps make the following action more entertaining. In continuum, the Liouville equation may be considered as a compatibility condition for the Liouville formula overloaded with Möbius-invariance. It ought to be just the same on the lattice, so let us find out what "equation of motion" the χ's might solve. We pick four of them next to each other to figure out that they are four cross-ratios made from the total of six α's and β's. That is one too many: 4 χ's + 3 symmetry parameters − 3 α's − 3 β's = 1. The missing one turns out to be

$$(7) \qquad \chi_{j,k+1}\chi_{j,k-1} = \frac{\chi_{j-1,k}\chi_{j+1,k}}{(1+\chi_{j-1,k})(1+\chi_{j+1,k})} \qquad j+k \equiv 0 \pmod 2$$

which so becomes our favourite lattice Liouville equation — replacing Hirota's original

$$h_{j,k+1}h_{j,k-1} = \frac{h_{j-1,k}h_{j+1,k}}{1+h_{j-1,k}h_{j+1,k}}.$$

The two are connected by a simple change of variables

$$\chi_{jk} = h_{j-1,k}h_{j+1,k},$$

which quarters the h's in the empty sites of our lattice of course. However, that change would also split cross-ratios, which we want to avoid, at least in this paper.

The stress-energy connection (3) also lends itself to a cross-ratio treatment. The Schwarz derivative inevitably becomes another cross-ratio [17]

$$u_n = 4\frac{(\alpha_{n+2}-\alpha_{n+1})(\alpha_n-\alpha_{n-1})}{(\alpha_{n+2}-\alpha_{n-1})(\alpha_{n+1}-\alpha_{n-1})},$$

then another counting argument leads to

$$(8) \qquad \frac{4}{u_{\frac{1}{2}(j-k)}} = \left(1+\chi_{j,k-1}+\frac{\chi_{j,k-1}}{\chi_{j-1,k}}\right)\left(1+\chi_{j+1,k}+\frac{\chi_{j+1,k}}{\chi_{j+2,k-1}}\right).$$

2.3. **Poisson bracket.** Two rows of χ's make perfect Cauchy data

$$\chi_{2n} = \chi_{2n,-1} \qquad\qquad \chi_{2n+1} = \chi_{2n+1,0},$$

just in case one may directly check that the lattice canonical bracket of [21]

$$\{\chi_{2n\pm1},\chi_{2n}\} = \chi_{2n\pm1}\chi_{2n} \qquad \{\chi_j,\chi_i\} = 0 \text{ if } |j-i| \not\equiv 1 \pmod N$$

both a) reproduces itself as the χ's evolve according to the lattice Liouville equation (7) and b) translates, by means of formula (8) at $k = 0$, into the lattice Magri bracket (5′).

3. Formal quantization

3.1. **Algebra of observables.** In a virtual replay of subsection 1.5, we introduce a formal quantization constant q, replace the above lattice canonical bracket by Weyl-style exchange relations

$$\chi_{2n\pm1}\chi_{2n} = q^2\chi_{2n}\chi_{2n\pm1} \qquad \chi_j\chi_i = \chi_i\chi_j \text{ if } |j-i| \not\equiv 1 \pmod{2N},$$

opt for even N and impose an additional condition on central elements

$$(9) \qquad \chi_1\chi_3^{-1}\chi_5\chi_7^{-1}\cdots\chi_{2N-3}\chi_{2N-1}^{-1} = \chi_2\chi_4^{-1}\chi_6\chi_8^{-1}\cdots\chi_{2N-2}\chi_{2N}^{-1}.$$

3.2. **Quantum lattice Liouville equation.** We can only afford a Heisenberg evolution of observables, so let us again consider χ_j as initial data ($\chi_{2n,-1} = \chi_{2n}$, $\chi_{2n+1,0} = \chi_{2n+1}$) and determine elements $\chi_{j,k+1}$, $j + k \equiv 0$ (mod 2) step by step using a slightly "quantized" lattice Liouville equation

$$(10) \qquad \chi_{j,k+1}\chi_{j,k-1} = \frac{q^2\chi_{j-1,k}\chi_{j+1,k}}{(1 + q\chi_{j-1,k})(1 + q\chi_{j+1,k})}.$$

We deliberately present the r.h.s. as a ratio to stress that all the factors there commute with each other, on the contrary, those in the l.h.s commute neither with each other nor with the r.h.s..

3.3. **Shift and evolution operators.** If we guessed the equation right, there must exist the "evolution operator", that is an element K such that

$$\chi_{j,k+1} = K^{-1}\chi_{j,k-1}K.$$

Needless to say, these relations will hold everywhere if and only if they do so for k equal 0 and -1 where they become

$$\chi_{2n-1}K\chi_{2n-1}K^{-1} = \frac{q^2\chi_{2n-2}\chi_{2n}}{(1 + q\chi_{2n-2})(1 + q\chi_{2n})}$$

$$K^{-1}\chi_{2n}K\chi_{2n} = \frac{q^2\chi_{2n-1}\chi_{2n+1}}{(1 + q\chi_{2n-1})(1 + q\chi_{2n+1})}.$$

Following [19, 21] we shall produce that element, with explicitly singled out d'Alembert part K_∞ good for

$$\chi_{2n-1}K_\infty\chi_{2n-1}K_\infty^{-1} = q^2\chi_{2n-2}\chi_{2n} \qquad K_\infty^{-1}\chi_{2n}K_\infty\chi_{2n} = q^2\chi_{2n-1}\chi_{2n+1},$$

and complete with the "shift operator" J moving the χ's in the spatial direction

$$\chi_{j+1,k} = J^{-1}\chi_{j-1,k}J.$$

Here follow explicit formulas for these shift and evolution operators

$$K_\infty = VU \qquad\qquad K = E_2K_\infty E_1 \qquad\qquad J = VU^{-1}$$

where

$$E_1 = \prod_{n=1}^{N}\epsilon(\chi_{2n-1}) \qquad\qquad E_2 = \prod_{n=1}^{N}\epsilon(\chi_{2n})$$

$$U = \theta(q\chi_1^{-1}\chi_2)\theta(q\chi_3^{-1}\chi_4)\ldots\theta(q\chi_{2N-3}^{-1}\chi_{2N-2})$$

$$V = \theta(q\chi_{2N-1}\chi_{2N-2}^{-1})\theta(q\chi_{2N-3}\chi_{2N-4}^{-1})\ldots\theta(q\chi_3\chi_2^{-1}).$$

Of course, U and V have everything to do with braids but we will not go into that. The two special functions involved

$$\epsilon(z) = \prod_{p=0}^{\infty}(1 + q^{2p+1}z) = (-qz; q^2)_\infty$$

$$\theta(z) = \epsilon(z)\,\epsilon(z^{-1}) = \mathrm{const}\cdot\sum_{p=-\infty}^{\infty}q^{p^2}z^p$$

are quite special indeed but all we want to know about them, for now at least, are the beautiful functional equations

$$\frac{\epsilon(qz)}{\epsilon(q^{-1}z)} = \frac{1}{1+z} \qquad\qquad \frac{\theta(qz)}{\theta(q^{-1}z)} = \frac{1}{z}$$

fulfiled by the former, and the ensuing equation on the latter which has already been used, albeit implicitly, in subsection 2.4. This time those functional equations and condition (9) gradually translate into relations

$$E_1^{-1}\chi_{2n-1}E_1 = \chi_{2n-1} \qquad E_1^{-1}\chi_{2n}E_1 = \chi_{2n}(1+q\chi_{2n-1})(1+q\chi_{2n+1})$$
$$E_2^{-1}\chi_{2n}E_2 = \chi_{2n} \qquad E_2^{-1}\chi_{2n-1}E_2 = ((1+q\chi_{2n-2})(1+q\chi_{2n}))^{-1}\chi_{2n-1}$$
$$U^{-1}\chi_{2n}U = \chi_{2n-1} \qquad U^{-1}\chi_{2n+1}U = q^2\chi_{2n-1}\chi_{2n+1}\chi_{2n}^{-1}$$
$$V^{-1}\chi_{2n}V = \chi_{2n+1} \qquad V^{-1}\chi_{2n-1}V = q^2\chi_{2n-1}\chi_{2n+1}\chi_{2n}^{-1}$$

which combined make J and K satisfy what they have to. We omit the calculation but a remark is in order.

3.4. Odd N or no condition (9).
In these cases it all fails, and subsection 1.6 gives an idea why. In fact, they are better served by those crossratio-splitting variables h and Hirota's lattice Liouville equation which we were so quick to discard. We are planning to return to this issue elsewhere.

3.5. Chiral evolution operator.
We define it as

$$Q = UE_1 \, ,$$

and it indeed moves the χ's in the right direction

$$\chi_{j,k+1} = Q^{-1}\chi_{j+1,k}Q$$

and equals $\sqrt{J^{-1}K}$, in the sense that $Q^2 = J^{-1}K$. So, the total momentum P, Hamiltonian H and the chiral Hamiltonian $\frac{1}{2}(H-P)$ of the original equation have finally become shift-evolution-operators J, K and Q of the quantum lattice equation, but we have yet to find out in what sense, if at all, this Q coincides with that hypothetical quantum lattice group-like counterpart of L_0 which we called Q in section 1.8. The answer does not come easy but in the end few carefully placed q's do it again, and so does the magic function ϵ introduced in subsection 3.3. It turns out that

(11) $$Q = d_1d_2\ldots d_{N-1},$$

where Q is UE_1 and the d's are

$$d_n = \epsilon\big(q^{-1}(1+q\chi_{2n}+\chi_{2n}\chi_{2n-1}^{-1})(1+q\chi_{2n+1}+\chi_{2n+2}^{-1}\chi_{2n+1})-q^{-1}\big).$$

As the notation suggests, these also satisfy relations (4″) and (5″). It is instructive now to see, that argument of function ϵ in this formula is a natural quantisation of expression (8).

3.6. **Proof of eqn (11).** First, we recall the two identities satisfied by ϵ-function:

(12) $\epsilon(u)\epsilon(v) = \epsilon(u+v), \quad \epsilon(v)\epsilon(u) = \epsilon(v+u+qvu), \quad uv = q^2 vu.$

Next, we have the identity

$$\epsilon(\chi_1(1+q\chi_2^{-1}))Q = Q\epsilon(\chi_{2N}^{-1}(1+q\chi_{2N-1}^{-1})^{-1})$$

which is a consequence of the relations in subsection 3.3 satisfied by U and E_1 operators. It is also easily checked that for any j

$$\theta(q\chi_{2j-1}^{-1}\chi_{2j})\epsilon(\chi_{2j-1}) = \epsilon(\chi_{2j-1}(1+q\chi_{2j}^{-1}))\epsilon((1+q\chi_{2j-1}^{-1})\chi_{2j}).$$

Now, using these relations, we have

$$Q = (\epsilon(\chi_1(1+q\chi_2^{-1})))^{-1} U E_1 \epsilon(\chi_{2N}^{-1}(1+q\chi_{2N-1}^{-1})^{-1})$$

$$= (\epsilon(\chi_1(1+q\chi_2^{-1})))^{-1} \left(\prod_{1\le j<N} \theta(q\chi_{2j-1}^{-1}\chi_{2j})\epsilon(\chi_{2j-1}) \right)$$

$$\times \epsilon(\chi_{2N-1})\epsilon(\chi_{2N}^{-1}(1+q\chi_{2N-1}^{-1})^{-1})$$

$$= (\epsilon(\chi_1(1+q\chi_2^{-1})))^{-1} \left(\prod_{1\le j<N} \epsilon(\chi_{2j-1}(1+q\chi_{2j}^{-1}))\epsilon((1+q\chi_{2j-1}^{-1})\chi_{2j}) \right)$$

$$\times \epsilon(q\chi_{2N-1}\chi_{2N}^{-1})\epsilon(\chi_{2N-1})$$

$$= \prod_{1\le j<N} \epsilon((1+q\chi_{2j-1}^{-1})\chi_{2j})\epsilon(\chi_{2j+1}(1+q\chi_{2j+2}^{-1})) = \prod_{1\le j<N} d_j,$$

thus obtaining eqn (11).

The proof of relations (5″) is done in [47] and is also based on identities (12).

4. Dualization

4.1. **Involution.** The formal quantization of section 3 can be put in the usual framework of quantum mechanics if we assume that both the formal parameter q and the generators χ are exponentials

$$q = e^{\pi i \tau} \qquad \chi_j = e^{-2\pi\sqrt{\tau}\varphi_j},$$

and the new generators φ_j have the commutation relations

$$[\varphi_{2n\pm1}, \varphi_{2n}] = -\frac{I}{2\pi i} \qquad [\varphi_j, \varphi_i] = 0 \text{ if } |j-i| \not\equiv 1 \pmod{N}$$

independent of τ, and so could be taken as selfadjoint operators

$$\varphi_j^\dagger = \varphi_j.$$

The rest depends on τ of course. If the established formula

$$c = 1 + 6(\tau + \tfrac{1}{\tau} + 2)$$

(relating the coupling constant of the continuous Liouville field theory to the central charge of the corresponding representation of the Virasoro algebra) has something to do with our lattice theory, then three cases have to be considered: a) $c \le 1 \leftrightarrow \tau < 0$; b) $c \ge 25 \leftrightarrow \tau > 0$; and c) $1 \le c \le 25 \leftrightarrow |\tau| = 1$. The first two, although lead to pretty normal reality conditions

$\chi_j^\dagger = \chi_j^{-1}$ and $\chi_j^\dagger = \chi_j$ respectively, also put q on the unit circle, which is the last thing we want right now. That leaves c).

4.2. Change of function ϵ.

So, $|\tau| = 1$, also, since c does not distinguish between τ and $1/\tau$, let for definiteness $\Im\tau > 0$. Following [13], consider the function

$$(13) \qquad f(z) = e_{\sqrt{\tau}}(iz/\sqrt{\tau}) = \frac{(-e^{2\pi i(z+\frac{\tau}{2})}; e^{2\pi i\tau})_\infty}{(-e^{\frac{2\pi i}{\tau}(z-\frac{1}{2})}; e^{-\frac{2\pi i}{\tau}})_\infty},$$

which uses our former favourite ϵ as the numerator but divides it by itself but with suitably altered arguments. In Appendix we collect some of the properties of this function. Here we remark in particular that $f(z)$ satisfies the same functional equation

$$(14) \qquad \frac{f(z + \frac{\tau}{2})}{f(z - \frac{\tau}{2})} = \frac{1}{1 + e^{2\pi i\tau z}}$$

as $\epsilon(e^{2\pi iz})$ did, but this time we also have

$$|f(z)| = 1$$

on the line $z = \tau\bar{z}$, simply because the numerator and denominator of f are complex conjugate of each other on that line. There is an easy profit to be had from that.

4.3. Unitarity.

Replace ϵ by f in every factor of every shift-evolution operator, all those little factors and big operators will at once become unitary but all the relations we have found them to satisfy will remain intact, for they rely only on that one and only functional equation. For instance, let us write out the so upgraded chiral evolution operator:

$$Q = \kappa^{2(N-1)} e^{\pi i(\varphi_1 - \varphi_2)^2} e^{\pi i(\varphi_3 - \varphi_4)^2} \dots e^{\pi i(\varphi_{2N-3} - \varphi_{2N-2})^2} \prod_{n=1}^{N} f(i\sqrt{\tau}\varphi_{2n-1}),$$

where it is already taken into account that

$$f(z)f(-z) = \kappa^2 e^{-\frac{\pi i}{\tau}z^2},$$

with $\kappa = f(0)$ of course.

Mission accomplished, our lattice Liouville model has finally turned from an algebraic fantasy into a quite material unitary theory. Moreover, we shall momentarily see that we get a new feature.

4.4. Dual Liouville equation.

Let us permute the factors in the l.h.s of the lattice Liouville equation (10)

$$\chi_{j,k-1}\chi_{j,k+1} = \chi_{j,k-1}\left(\chi_{j,k+1}\chi_{j,k-1}\right)\chi_{j,k-1}^{-1}$$

$$= \chi_{j,k-1}\left(\frac{q^2\chi_{j-1,k}\chi_{j+1,k}}{(1 + q\chi_{j-1,k})(1 + q\chi_{j+1,k})}\right)\chi_{j,k-1}^{-1}$$

$$= \frac{q^{-2}\chi_{j-1,k}\chi_{j+1,k}}{(1 + q^{-1}\chi_{j-1,k})(1 + q^{-1}\chi_{j+1,k})}$$

and treat the both sides with Hermitean conjugation, remembering that $Q^\dagger = Q^{-1}$ of course. Since

$$\tilde{q} \equiv \overline{q^{-1}} = e^{\frac{\pi i}{\tau}} = q^{(\frac{1}{\tau})^2} \qquad\qquad \tilde{\chi}_j \equiv \chi_j^\dagger = e^{-\frac{2\pi}{\sqrt{\tau}}\varphi_j} = \chi_j^{\frac{1}{\tau}},$$

the resulting equation reads

$$(10^*) \qquad\qquad \tilde{\chi}_{j,k+1}\tilde{\chi}_{j,k-1} = \frac{\tilde{q}^2 \tilde{\chi}_{j-1,k}\tilde{\chi}_{j+1,k}}{(1+\tilde{q}\tilde{\chi}_{j-1,k})(1+\tilde{q}\tilde{\chi}_{j+1,k})}.$$

It is plain to see that it is the same equation but with q and χ's replaced by $\tilde{q}$ and $\tilde{\chi}$'s, this is called duality. Indeed, instead of conjugating things, we might instead explore the fact that those $\tilde{\chi}$'s satisfy the same relations

$$\tilde{\chi}_{2n\pm1}\tilde{\chi}_{2n} = \tilde{q}^2 \tilde{\chi}_{2n}\tilde{\chi}_{2n\pm1}$$

as the original χ's do, except with $\tilde{q}$ instead of q. Moreover, the two sets commute with each other

$$\chi_j \tilde{\chi}_i = \tilde{\chi}_i \chi_j \ .$$

The algebras they generate (call them $\mathcal{A}_\tau$ and $\mathcal{A}_{\frac{1}{\tau}}$) form two factors in the algebra $\mathcal{B}$ generated by the φ's, and leave no free space, in the sense that $\mathcal{B} = \mathcal{A}_\tau \otimes \mathcal{A}_{\frac{1}{\tau}}$, sort of. Either way, such a bisection is well served by the function f, which fittingly satisfies the dual functional equation

$$(14^*) \qquad\qquad \frac{f(z+\frac{1}{2})}{f(z-\frac{1}{2})} = \frac{1}{1+e^{\frac{2\pi i}{\tau}z}}$$

to complement the original one (14). Equation (10^*) can now be derived in exactly the same way as (10) was, and regarded as not just a conjugate clown of the latter but as an equal dual equation.

4.5. Baxter equation. It reads

$$t_\tau(\lambda)Q(\lambda) = Q(\lambda+\tfrac{1}{2}) + (e^{4\pi i\tau\lambda}+1)^N Q(\lambda-\tfrac{1}{2}),$$

where $t_\tau(\lambda)$ and $Q(\lambda)$ are two families of elements (of $\mathcal{A}_\tau$ and $\mathcal{B}$ respectively), which all commute with each other

$$[t_\tau(\lambda), t_\tau(\mu)] = [Q(\lambda), t_\tau(\mu)] = [Q(\lambda), Q(\mu)].$$

In this paper we do not define those families explicitly and do not derive the equation. Let us only mention that the chiral evolution operator Q is in fact $Q(\lambda)$ evaluated at a particular value of λ. The duality symmetry of our construction implies that there is another dual equation having the form

$$t_{\frac{1}{\tau}}(\lambda)Q(\lambda) = Q(\lambda+\tfrac{\tau}{2}) + (e^{4\pi i\lambda}+1)^N Q(\lambda-\tfrac{\tau}{2}).$$

Thus, the Baxter equation gets accociated with the modular lattice.

We stop here and postpone the study of Baxter equations to the next paper of this series.

5. Conclusions and comments

We have shown, that the appropriate lattice regularization of the quantum Liouville model allows to enter the "forbidden region" of the Virasoro central charge $1 \leq c \leq 25$. The key for this is using the double family of dual quantum fields, which are not selfadjoint, but normal and adjoint to each other. The results are still rather modest, but proving the point quite persuasively.

The real problem is that of findng the spectrum of the model. One viable approach to this can be based on the Baxter equation. In the next paper of this series we plan to deal with this equation. The idea of its derivation originally goes to Baxter himself [2]. In [8, 4] it was realized in the context of the chiral Potts model, and in [5, 6, 7] in the context of quantum conformal field theory. The dual Baxter equations in a different context were also considered recently by Smirnov [39].

In parallel construction for the WZNW model [12] it was shown how to separate the zero mode problem which reduces the problem of finding the spectrum of the primary states to a problem with finite number of degrees of freedom. It is possible that similar construction can be found here. If so, one will be able to bypass Baxter equations for the investigation of the spectrum.

Liouville model is known to be a contraction of the massive Sine–(Sinh–) Cordon model to which most of our considerations are also applicable [20].

In a series of papers [5, 6, 7] the quantum KdV equation was considered without recourse to lattice. As the Liouville and KdV models are close relatives, it is no wonder that one can see many similarities between those papers and our text. Moreover, the discrete variant of quantum KdV was already considered in [46]. We believe, that using lattice allows to use the duality in full strength and, in particular, go to the strong coupling region.

In papers [27, 28] arguments were raised for the exceptional values of the central charge $c = 7$, 13, 19. From our point of view, these values are distinguished only by giving our modular lattice supplementary symmetries (τ corresponds to elliptic fixed points of the modular figure: $e^{i\pi/3}$, $e^{i\pi/2}$, $e^{i2\pi/3}$). There is no reason for us to exclude other values of τ on the unit half-circle.

Finally, we cannot help stating that for us the function $f(z)$ defined in eqn (13) seems to be the real cornerstone of the theory of quantum integrable models. It appears in basic objects of the dynamical theory: the evolution and the Baxter operators. Its close relative as a function of rapidity defines the Zamolodchikov's factorized S-matrices. It is also indispensable in the theory of form-factors [38]. We believe that the full content of duality, hidden in this function, is still not explored to full extent. In the Appendix below we describe some of its remarkable properties.

6. Appendix: the non-compact quantum dilogarithm

Let complex b have a nonzero real part $\Re b \neq 0$. The non-compact QDL, $e_b(z)$, $z \in \mathbb{C}$, $|\Im z| < |\Im c_b|$,

$$c_b \equiv i(b + b^{-1})/2,$$

is defined by the formula

$$(15) \qquad e_b(z) \equiv \exp\left(\frac{1}{4}\int_{-\infty}^{+\infty}\frac{e^{-2izx}\,dx}{\sinh(xb)\sinh(x/b)x}\right),$$

where the singularity at $x = 0$ is put below the contour of integration. This definition implies that $e_b(z)$ is unchanged under substitutions $b \to b^{-1}$, $b \to -b$. Using this symmetry, we choose b to lay in the first quadrant of the complex plane, namely

$$\Re b > 0, \quad \Im b \geq 0,$$

which implies that

$$\Im \tau > 0, \quad \tau \equiv b^2.$$

This function has the following properties.

6.1. **Functional relations.** Function (15) satisfies the 'inversion' relation

$$(16) \qquad e_b(z)e_b(-z) = e^{i\pi z^2 - i\pi(1+2c_b^2)/6},$$

and a pair of functional equations

$$(17) \qquad e_b(z - ib^{\pm 1}/2) = (1 + e^{2\pi zb^{\pm 1}})e_b(z + ib^{\pm 1}/2).$$

The latter equations enable us to extend the definition of the QDL to the entire complex plane.

When b is real or a pure phase, function $e_b(z)$ is *unitary* in the sense that

$$(18) \qquad \overline{e_b(z)} = 1/e_b(\bar{z}).$$

If selfadjoint operators P and X in $L^2(\mathbb{R})$ satisfy the Heisenberg commutation relations

$$(19) \qquad [\mathsf{P}, \mathsf{X}] = \frac{1}{2\pi i},$$

the following operator five term identity holds:

$$(20) \qquad e_b(\mathsf{P})e_b(\mathsf{X}) = e_b(\mathsf{X})e_b(\mathsf{P} + \mathsf{X})e_b(\mathsf{P}).$$

For real b this can be proved in the C^*-algebraic framework [1]. We will prove it for complex b. The case of real b then will follow by continuity.

6.2. **Analytic properties.** We can perform the integration in (15) by the residue method. The result can be written as ratio of two q-exponentials

$$(21) \qquad e_b(z) = (e^{2\pi(z+c_b)b}; q^2)_\infty / (e^{2\pi(z-c_b)b^{-1}}; \bar{q}^2)_\infty,$$

where

$$q = e^{i\pi b^2}, \quad \bar{q} = e^{-i\pi b^{-2}},$$

thus reprpoducing definition (13) in the text. Formula (21) defines a meromorphic function on the entire complex plane, satisfying functional equations (16) and (17), with essential singularity at infinity. So, it is the analytical continuation of definition (15) to the entire complex plane. It is easy to read off location of its poles and zeroes:

$$\text{zeroes of } (e_b(z))^{\pm 1} = \{\mp(c_b + mib + nib^{-1}) : \ m, n \in \mathbb{Z}_{\geq 0}\}.$$

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[1]S.L. Woronowicz: private communication, 1998

The behavior at infinity depends on the direction along which the limit is taken:

$$(22) \quad e_b(z)\Big|_{|z|\to\infty} \approx \begin{cases} 1 & |\arg(z)| > \frac{\pi}{2} + \arg(b); \\ e^{i\pi z^2 - i\pi(1+2c_b^2)/6} & |\arg(z)| < \frac{\pi}{2} - \arg(b); \\ (\bar{q}^2; \bar{q}^2)_\infty / \Theta(ib^{-1}z; -b^{-2}) & |\arg z - \pi/2| < \arg b; \\ \Theta(ibz; b^2)/(q^2; q^2)_\infty & |\arg z + \pi/2| < \arg b, \end{cases}$$

where

$$\Theta(z; \tau) \equiv \sum_{n \in \mathbb{Z}} e^{i\pi\tau n^2 + 2\pi inz}, \quad \Im\tau > 0.$$

Thus, for complex b, double quasi-periodic θ-functions, generators of the field of meromorphic functions on complex tori, describe the asymptotic behavior of the non-compact QDL.

6.3. **Integral Ramanujan identity.** Consider the following Fourier integral:

$$(23) \qquad \Psi(u, v, w) \equiv \int_{\mathbb{R}} \frac{e_b(x + u)}{e_b(x + v)} e^{2\pi iwx}\, dx,$$

where

$$(24) \qquad \Im(v + c_b) > 0, \quad \Im(-u + c_b) > 0, \quad \Im(v - u) < \Im w < 0.$$

Restrictions (24) actually can be considerably relaxed by deforming the integration path in the complex x plane, keeping the asymptotic directions of the two ends within the sectors $\pm(|\arg x| - \pi/2) > \arg b$. So, the enlarged in this way domain for the variables u, v, w has the form:

$$(25) \qquad |\arg(iz)| < \pi - \arg b, \quad z \in \{w, v - u - w, u - v - 2c_b\}.$$

Regarding $e_b(z)$ as a 'non-compact' analogue of the q-exponent $(x; q)_\infty$, definition (23) can be interpreted as the corresponding integral counterpart of the Ramanujan sum:

$$_1\psi_1(x, y, z) \equiv \sum_{n \in \mathbb{Z}} \frac{(x; q)_n}{(y; q)_n} z^n.$$

The latter is known to be evaluated explicitly, the result being the famous Ramanujan summation formula:

$$_1\psi_1(x, y, z) = \frac{(q; q)_\infty (y/x; q)_\infty (xz; q)_\infty (q/xz; q)_\infty}{(y; q)_\infty (q/x; q)_\infty (z; q)_\infty (y/xz; q)_\infty}.$$

Remarkably, integral (23) can be evaluated explicitly as well. Indeed, using the residue method, we easily come to the following result:

$$(26) \qquad \Psi(u, v, w) = \frac{e_b(u - v - c_b)e_b(w + c_b)}{e_b(u - v + w - c_b)} e^{-2\pi iw(v+c_b)+i\pi(1-4c_b^2)/12}$$

$$(27) \qquad = \frac{e_b(v - u - w + c_b)}{e_b(v - u + c_b)e_b(-w - c_b)} e^{-2\pi iw(u-c_b)-i\pi(1-4c_b^2)/12},$$

where the two expressions in the right hand side are related to each other through the inversion relation (16). The similarity of this result with Ramanujan sum becomes very transparent if we rewrite the latter in the form:

$$(28) \qquad \sum_{n \in \mathbb{Z}} \frac{(yq^n; q)_\infty}{(xq^n; q)_\infty} z^n = \frac{(y/x; q)_\infty (q/z; q)_\infty}{(y/xz; q)_\infty} \frac{\theta_q(xz)}{\theta_q(x)\theta_q(z)} (q; q)_\infty^2,$$

where the θ_q-function is defined by

$$(29) \qquad \theta_q(x) \equiv (q; q)_\infty (x; q)_\infty (q/x; q)_\infty = \sum_{n \in \mathbb{Z}} q^{n(n-1)/2}(-x)^n.$$

Comparing the inversion relation (16) with eqn (29), we conclude that the non-compact analogue of the θ-function is the Gaussian exponent, and the structures of eqns (28) and (26) are now quite similar.

6.4. Fourier transformation of the QDL.

Certain specializations of $\Psi(u, v, w)$ lead to the following Fourier transformation formulas for the QDL:

$$(30) \quad \phi_+(w) \equiv \int_{\mathbb{R}} e_b(x) e^{2\pi i w x} \, dx = \Psi(0, v, w)|_{v \to -\infty}$$

$$= e^{2\pi i w c_b - i\pi(1-4c_b^2)/12} / e_b(-w - c_b) = e^{-i\pi w^2 + i\pi(1-4c_b^2)/12} e_b(w + c_b),$$

and

$$(31) \quad \phi_-(w) \equiv \int_{\mathbb{R}} (e_b(x))^{-1} e^{2\pi i w x} \, dx = \Psi(u, 0, w)|_{u \to -\infty}$$

$$= e^{-2\pi i w c_b + i\pi(1-4c_b^2)/12} e_b(w + c_b) = e^{i\pi w^2 - i\pi(1-4c_b^2)/12} / e_b(-w - c_b),$$

The corresponding inverse transformations read:

$$(32) \qquad\qquad (e_b(x))^{\pm 1} = \int_{\mathbb{R}} \phi_\pm(y) e^{-2\pi i x y} dy,$$

where the pole at $y = 0$ is surrounded from below.

6.5. Proof of the Pentagon identity.

Using formula (32) and commutation relation (19), we equate the coefficients of the operator terms

$$e^{-2\pi i x \mathsf{X}} e^{-2\pi i y \mathsf{P}}$$

in the pentagon relation (20), the result being an integral identity:

$$\phi_+(x)\phi_+(y) e^{2\pi i x y} = \int_{\mathbb{R}} \phi_+(z)\phi_+(x - z)\phi_+(y - z) e^{i\pi z^2} dz,$$

where the singularities at $z = x$, $z = y$ are put below, and at $z = 0$, above the integration path. Now, multiplying both sides of this identity by $\exp(-2\pi i y u)$, integrating over y, and using (32), we obtain

$$\phi_+(x) e_b(u - x) = e_b(u) \int_{\mathbb{R}} \phi_+(z)\phi_+(x - z) e^{i\pi z^2 - 2\pi i u z} dz.$$

Using (30), we rewrite it equivalently

$$(33) \quad \frac{e_b(u - x)}{e_b(-x - c_b) e_b(u)} e^{-i\pi(1-4c_b^2)/12}$$

$$= \int_{\mathbb{R}} \frac{e_b(z + c_b)}{e_b(z - x - c_b)} e^{-2\pi i z(u + c_b)} dz = \Psi(c_b, -x - c_b, -u - c_b),$$

which is a particular case of (27).

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